EE103 Lect #8 Oct 16, 2017

Quiz1 Average = 7.16/10.0 $\sigma = 2.02$ (88 students)

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{+\infty}^{-\infty} x(t-\tau)\, h(\tau)\, (-d\tau)$$

$\tau \leftarrow t-\tau$

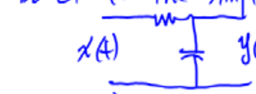
$d\tau \leftarrow -d\tau$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau = h(t) * x(t)$$

In summary

$$\boxed{\begin{aligned} x(t) * h(t) &= h(t) * x(t) \\ f_1(t) * f_2(t) &= f_2(t) * f_1(t) \end{aligned}}$$

Back to the simple RC circuit



$\delta(t) \xrightarrow{\mathcal{H}} h(t)$

$h(t)\ \frac{1}{RC} u(t) e^{-\frac{1}{RC}t}$

what is $y(t)$ for $x(t) = u(t)$?

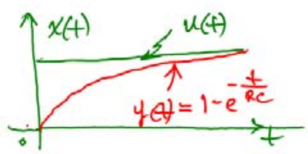
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} \frac{u(\tau)}{RC} e^{-\frac{1}{RC}(t-\tau)} \cdot u(t-\tau)\, d\tau$$

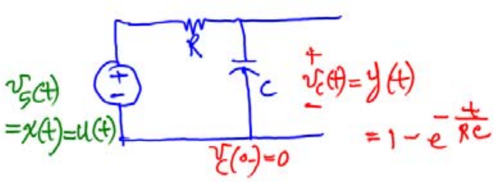
$$= \int_{-\infty}^{\infty} u(t-\tau) \frac{e^{-\frac{1}{RC}\tau}}{RC} u(\tau)\, d\tau$$

$$= \int_0^t \frac{1}{RC} e^{-\frac{1}{RC}\tau}\, d\tau$$

$$= \int_0^t d\left[-e^{-\frac{1}{RC}\tau}\right]$$

$$= \left(1 - e^{-\frac{1}{RC}t}\right) = y(t)$$





Convolution of periodic signal

2.7. Let $h(t)$ be the triangular pulse shown in Fig. 2-10(a) and let $x(t)$ be the unit impulse train [Fig. 2-10(b)] expressed as

$$x(t) = \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad (2.68)$$

Determine and sketch $y(t) = h(t) * x(t)$ for the following values of T: (a) $T = 3$, (b) $T = 2$, (c) $T = 1.5$.

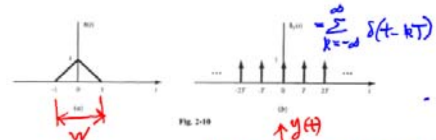


Fig. 2-10

Properties of Convolution

$x(t) \rightarrow \boxed{\mathcal{H}} \rightarrow y(t)$

$\delta(t) \xrightarrow{\mathcal{H}} h(t)$ impulse response

$x(t) \xrightarrow{\mathcal{H}} y(t) = x(t) * h(t)$

Property 1 (Commutative)

$$x(t) * h(t) = h(t) * x(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Property 2 (Associative)

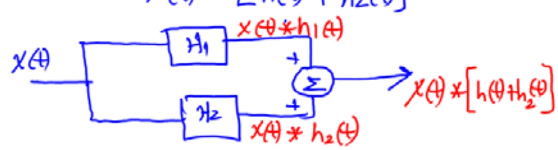
$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

$x(t) \rightarrow \boxed{\mathcal{H}_1} \xrightarrow{x * h_1} \boxed{\mathcal{H}_2} \xrightarrow{(x * h_1) * h_2}$ ||

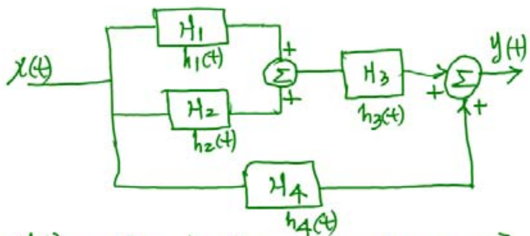
$x(t) \rightarrow \boxed{\mathcal{H}_S} \rightarrow x(t) * (h_1(t) * h_2(t))$

Property 3 (Distributive)

$$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * [h_1(t) + h_2(t)]$$



$x(t) \rightarrow \boxed{\mathcal{H}_P} \rightarrow x(t) * [h_1(t) + h_2(t)]$



$$y(t) = x(t) * h_4(t) + \left[x(t) * (h_1(t) + h_2(t)) \right] * h_3(t)$$
$$= x(t) * \left[(h_1(t) + h_2(t)) * h_3(t) + h_4(t) \right]$$

General Form of n_{th}-order Linear (ordinary) differential equation

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

where $a_k,\ k=0,n$ and $b_j,\ j=0,m$ are constants

More compact form

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{j=0}^{m} b_j \frac{d^j x(t)}{dt^j} \quad (\mathrm{I})$$

solution $\boxed{y(t) = y_c(t) + y_p(t)}$

$y_c(t)$ = complementary solution (natural response)

$y_p(t)$ = particular solution (forced response)

Natural response $= y(t)\big|_{\text{when } x(t)=0}$, i.e. zero input

From (A) $\displaystyle\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = 0 \quad (\mathrm{II})$

In general $y_c(t) = C e^{st} \quad (\mathrm{III})$

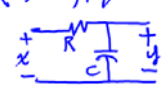
From (II) & (III), we obtain

$$\left(\sum_{k=0}^{n} a_k s^k \right) C e^{st} = 0 \quad \Rightarrow$$

$$\Rightarrow \quad a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$
$$a_n (s - s_1)(s - s_2) \cdots (s - s_n) = 0 \quad (\mathrm{IV})$$

then $y_c(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \cdots + C_n e^{s_n t}$

(Example)



$$C \frac{dy(t)}{dt} = \frac{x(t) - y(t)}{R}$$
$$RC \frac{dy(t)}{dt} + y(t) = x(t)$$
$$a_1 \frac{d^1 y(t)}{dt^1} + a_0 y(t) = x(t)$$
$$a_1 = RC,\ a_0 = 1,\ b_0 = 1$$

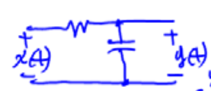
$y_c(t) = ?$ set $x(t) = 0$

$$RC \frac{dy}{dt} + y = 0$$

Let $y_c(t) = c_1 e^{s_1 t}$

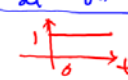
$$RC \frac{d}{dt}(c_1 e^{s_1 t}) + c_1 e^{s_1 t} = 0$$
$$RC\, c_1 s_1 e^{s_1 t} + c_1 e^{s_1 t}$$
$$= (RC s_1 + 1) c_1 e^{s_1 t} = 0$$
$$\Rightarrow \boxed{RC s_1 + 1 = 0} \Rightarrow \boxed{s_1 = -\frac{1}{RC}}$$

$y_c(t) = c_1 e^{-\frac{1}{RC}t}$



$y(0) = 0$ $\quad \boxed{RC \frac{dy(t)}{dt} + y(t) = x(t)}$

For $x(t) = u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$



$$RC \frac{dy_p(t)}{dt} + y_p(t) = 1,\ t > 0$$

$y_p(t) = P$ (constant), $\frac{dy_p}{dt} = 0$

$y_p(t) = P = 1$

$$y(t) = y_c(t) + y_p(t) = c_1 e^{-\frac{1}{RC}t} + 1$$
$$y(0) = 0 = c_1 e^{-\frac{1}{RC}(0)} + 1 \Rightarrow c_1 = -1$$
$$\Rightarrow \boxed{y(t) = 1 - e^{-\frac{1}{RC}t}}$$

Revisit of BIBO stability

$y(t) = \mathcal{H}\, x(t)$

$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

Taking absolute value of both sides

$|y(t)| = \left| \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right|$

$\leq \int_{-\infty}^{\infty} |x(\tau)|\, |h(t-\tau)|\, d\tau$

$\boxed{|y(t)| \leq N \text{ for all } |x(t)| \leq M \quad (\text{B.I.B.O}) \text{ if and only if } \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty}$

- $h(t) = A e^{-kt}, k>0$ BIBO
- $h(t) = u(t)$ Integrator, not BIBO
- $h(t) = R(t)$ not BIBO
- $h(t) = \mathrm{rect}(t - \frac{T}{2} | T)$ BIBO
- $h(t) = \sum_{k=0}^{\infty} \delta(t-kT)$ not BIBO